2

VOL. 104 NO. HY10. OCT. 1978

JOURNAL OF THE HYDRAULICS DIVISION

PROCEEDINGS OF THE AMERICAN SOCIETY OF CIVIL ENGINEERS

JOURNAL OF THE
HYDRAULICS DIVISION

Analytic Solutions for Computer Flow Model Testing

By Daniel R. Lynch[1] and William G. Gray,[2] A. M. ASCE

Introduction

Various numerical methods for the solution of the shallow water equations have been applied to problems of flood routing, tidal circulation, storm surges, and atmospheric circulation (1,3,4,5,8,10,12,14). Utility of such methods is often demonstrated by comparison of computed variables with field observations. However this type of comparison is often incapable of adequately verifying that a numerical model accurately represents the dynamics of the study region. The limitations of this approach are due to inadequate data and incomplete understanding of the behavior of the numerical procedure.

Observations of water depth and velocity are rarely available throughout the temporal and spatial domains of interest. While depth measurements are common near shoreline boundaries, they are much less common and more difficult to obtain in the open sea. Accurate vertically or cross-sectionally averaged velocity data are even more scarce. Experimentally undetected subregions may also exist in which the shallow water equations do not accurately describe the flow phenomenon. Thus, in general, data bases are inadequate tools for establishing that a numerical model is correctly solving the governing equations.

The precision with which a numerical scheme solves the full governing equations should also be established. Because of the nonlinearities in the equations, this is difficult to ascertain precisely. Furthermore the effect of an irregularly shaped boundary on the accuracy of the numerical solution is generally not completely known although it is acknowledged to be important.

These various sources of error and uncertainty in verification can many times be completely buried in the numerical solution by adjustment of parameters such as bathymetry, eddy viscosity, and Chezy coefficients. It is our belief

Note.—Discussion open until March 1, 1979. To extend the closing date one month, a written request must be filed with the Editor of Technical Publications, ASCE. This paper is part of the copyrighted Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 104, No. HY10, October, 1978. Manuscript was submitted for review for possible publication on February 19, 1978.

[1]Asst. Prof., Thayer School of Engrg., Dartmouth Coll., Hanover, N.H.

[2]Asst. Prof., Water Resources Program, Dept. of Civ. Engrg., Princeton Univ., Princeton, N.J.

that a more systematic and rigorous assessment of error sources must be made in order to establish the credibility of a numerical model. To this end a number of analytic solutions are herein developed which should prove useful for comparison to numerical solutions. By necessity the shallow water equations have been linearized. However bottom friction, wind stress, Cartesian and polar geometry, and variable bathymetry have been incorporated into the equations. Solutions for the dynamic steady state with a periodic forcing function and for startup from rest are obtained. In line with the philosophy that these solutions are useful primarily as tools for model verification, emphasis is placed on the periodic solutions.

Basic Equations

The linearized shallow water equations will be solved in the subsequent examples. These equations are obtained from the full shallow water equations by neglecting the convective terms, assuming the oscillations of the free surface are small in comparison to the total depth, and using a linearized friction term. Thus the equations take the form:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [h\,\mathbf{v}] = 0 \quad \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + g\nabla\zeta + \tau\mathbf{v} - \frac{\mathbf{W}}{h} = 0 \quad \quad (2)$$

in which $\zeta(x,y,t)$ = the free surface elevation above mean sea level; $\mathbf{v}(x,y,t)$ = the vertically averaged fluid velocity; $h(x,y)$ = the vertical distance from mean sea level to the floor of the water body; g = gravity; τ = the linearized bed friction parameter assumed constant; $\mathbf{W}$ = the wind stress, assumed to be spatially invariant; t = time; and x,y = the horizontal spatial coordinates.

Differentiation of Eq. 1 with respect to time yields

$$\frac{\partial^2 \zeta}{\partial t^2} + \nabla \cdot \left[h\frac{\partial \mathbf{v}}{\partial t} \right] = 0 \quad \quad (3)$$

Substitution of Eq. 2 into Eq. 3 for $\partial\mathbf{v}/\partial t$ and rearrangement yields

$$\frac{\partial^2 \zeta}{\partial t^2} - \nabla\cdot(gh\nabla\zeta) - \tau\nabla\cdot(h\mathbf{v}) = 0 \quad \quad (4)$$

Finally, substitution of Eq. 1 into Eq. 4 for $\nabla\cdot(h\mathbf{v})$ yields

$$\frac{\partial^2 \zeta}{\partial t^2} + \tau\frac{\partial \zeta}{\partial t} - g\nabla\cdot(h\nabla\zeta) = 0 \quad \quad (5)$$

This equation, together with appropriate boundary conditions, will be solved for ζ in the subsequent examples. The solution for $\mathbf{v}$ is then obtained from

Periodic Steady State Solutions

Case I: Polar Geometry.—The problem considered here is shown in Fig. 1. Flow is required to be tangent to the solid boundaries at $r = r_1$, $\theta = 0$ and $\theta = \phi$. A tidal forcing function is specified at $r = r_2$, and constant wind stress is imposed throughout, in an arbitrary direction. Bathymetry is described by $h = H_o r^n$ in which H_o is a constant, and n is not necessarily an integer and may assume any real value. Boundary conditions are:

$$\text{at} \quad r = r_1: \quad \frac{\partial \zeta}{\partial r} - \frac{W_r}{gh} = 0 \quad \text{(6a)}$$

$$\text{at} \quad r = r_2: \quad \zeta(r_2, \theta, t) = \text{Re}\,\{\zeta_o(\theta) e^{i\omega t}\} \quad \text{(6b)}$$

$$\text{at} \quad \theta = 0, \phi: \quad \frac{1}{r}\frac{\partial \zeta}{\partial \theta} - \frac{W_\theta}{gh} = 0 \quad \text{(6c)}$$

in which W_r, W_θ = the steady uniform wind stress components in the r and θ directions, respectively; ω = the frequency of the tidal forcing function; $\zeta_o(\theta)$

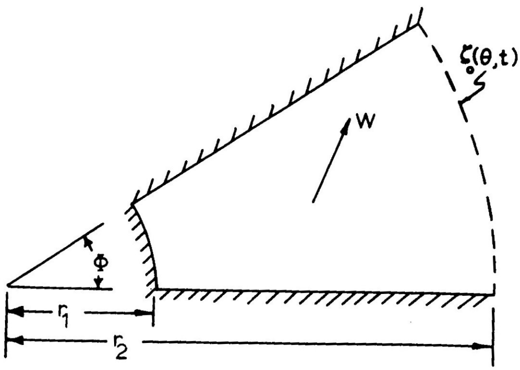


FIG. 1.—Annular Section in r-θ Coordinates with Opening at $r = r_2$

= a complex function representing the tidal amplitude and phase at $r = r_2$; and $i = \sqrt{-1}$.

The solution to Eq. 5 subject to Eq. 6 is most easily obtained by letting

$$\zeta = \zeta_f + \zeta_W \quad \text{(7)}$$

in which ζ_f = the solution to Eqs. 5 and 6 in the absence of wind stress; and ζ_W = the solution in the absence of a tidal forcing function. Thus two sets of equations must be solved:

$$r^2 \frac{\partial^2 \zeta_W}{\partial r^2} + r(1+n)\frac{\partial \zeta_W}{\partial r} + \frac{\partial^2 \zeta_W}{\partial \theta^2} = 0 \quad \text{(8a)}$$

$$\text{at} \quad r = r_1: \quad \frac{\partial \zeta_W}{\partial r} - \frac{W_r}{g H_o r^n} = 0 \quad \text{(8b)}$$

$$\text{at} \quad r = r_2: \quad \zeta_W(r_2, \theta) = 0 \quad \text{(8c)}$$

at $\theta = 0, \phi$: $$\frac{1}{r}\frac{\partial \zeta_W}{\partial \theta} - \frac{W_\theta}{gH_o r^n} = 0 \quad (8d)$$

and

$$\frac{\partial^2 \zeta_f}{\partial t^2} + \tau \frac{\partial \zeta_f}{\partial t} - gH_o r^{n-2}\left[r^2 \frac{\partial^2 \zeta_f}{\partial r^2} + r(n+1)\frac{\partial \zeta_f}{\partial r} + \frac{\partial^2 \zeta_f}{\partial \theta^2}\right] = 0 \quad (9a)$$

at $r = r_1$: $$\frac{\partial \zeta_f}{\partial r} = 0 \quad (9b)$$

at $r = r_2$: $$\zeta_f(r_2, \theta, t) = \mathrm{Re}\{\zeta_o(\theta)e^{i\omega t}\} \quad (9c)$$

at $\theta = 0, \phi$: $$\frac{\partial \zeta_f}{\partial \theta} = 0 \quad (9d)$$

Steady State Wind Setup

To solve Eqs. 8, consider first the case of unit wind stress in the direction $\theta = 0$. Assuming a solution of the form $\zeta_W(r,\theta) = R(r)\,T(\theta)$ in Eq. 8*a* results in

$$\frac{1}{R}\left[r^2 R'' + (1+n)rR'\right] = \kappa^2 \quad (10a)$$

$$\kappa^2 = -\frac{T''}{T} \quad (10b)$$

in which κ^2 = a separation constant. The general solution to Eq. 10 is

$$\zeta_W(r,\theta) = \sum_j (a_j r^{s_j} + b_j r^{t_j})[\cos(\kappa_j \theta) + c_j \sin(\kappa_j \theta)] \quad (11)$$

in which $$s_j = -\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \kappa_j^2} \quad (12a)$$

$$t_j = -\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \kappa_j^2} \quad (12b)$$

and $\sin(\kappa_j \theta)$ is replaced by θ when $\kappa_j = 0$. Boundary conditions for this problem are:

at $r = r_1$: $$\frac{\partial \zeta_W}{\partial r} = \frac{\cos\theta}{gH_o r_1^n} \quad (13a)$$

at $r = r_2$: $$\zeta_W = 0 \quad (13b)$$

at $\theta = 0$: $$\frac{\partial \zeta_W}{\partial \theta} = 0 \quad (13c)$$

at $\theta = \phi$: $$\frac{1}{r}\frac{\partial \zeta_W}{\partial \theta} = -\frac{\sin\phi}{gH_o r^n} \quad (13d)$$

Eq. 13c is satisfied by letting $c_j = 0$ for all j. Eq. 13d will be satisfied by a single component $\kappa_j = \kappa^*$ such that

$$s^* = 1 - n; \quad \kappa^* = (1-n)^{1/2}; \quad a^* = \frac{\sin \phi}{gH_o \kappa^* \sin(\kappa^* \theta)} \quad \text{. . . . . (14)}$$

This method fails for combinations of n and ϕ such that $\kappa^* = j\,\pi/\phi$, $j = 0, 1, 2, \ldots$. In these cases, the method described subsequently for Cartesian geometry must be used. Boundary conditions Eqs. 13c and 13d are unaffected by additional components in Eq. 11 when $\kappa_j = j\pi/\phi$, $j = 0, 1, 2, \ldots$. Thus, the most general solution which satisfies these two boundary conditions is

$$\zeta_W(r,\theta) = a^* r^{1-n} [\cos (1-n)^{1/2}\theta]$$

$$+ \sum_{j=0}^{\infty} (a_j r^{s_j} + b_j r^{t_j}) \cos\left(\frac{j\pi\theta}{\phi}\right) \quad \text{. . . . . (15a)}$$

in which $s_j = -\dfrac{n}{2} + \sqrt{\left(\dfrac{n}{2}\right)^2 + \left(\dfrac{j\pi}{\phi}\right)^2}$;

$$t_j = -\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{j\pi}{\phi}\right)^2} \quad \text{. . . . . (15b)}$$

To satisfy Eqs. 13a and 13b, first express $\cos\theta$ and $\cos[(1-n)^{1/2}\theta]$ as Fourier series in the interval $0 \le \theta \le \phi$:

$$\cos[(1-n)^{1/2}\theta] = D_o + \sum_{j=1}^{\infty} D_j \cos\left(\frac{j\pi\theta}{\phi}\right) \quad \text{. . . . . (16a)}$$

$$\cos\theta = E_o + \sum_{j=1}^{\infty} E_j \cos\left(\frac{j\pi\theta}{\phi}\right) \quad \text{. . . . . (16b)}$$

in which $D_o = \dfrac{\sin[(1-n)^{1/2}\phi]}{(1-n)^{1/2}\phi}$

$$D_j = \frac{2(-1)^j (1-n)^{1/2}\phi \sin[(1-n)^{1/2}\phi]}{(1-n)\phi^2 - j^2\pi^2} \quad \text{. . . . . (16c)}$$

$$E_o = \frac{\sin\phi}{\phi} \quad E_j = \frac{2(-1)^j \phi \sin\phi}{\phi^2 - j^2\pi^2} \quad \text{. . . . . (16d)}$$

Application of boundary conditions Eqs. 13c and 13d now gives a determination of a_j and b_j:

$$s_j a_j r_1^{s_j} + t_j b_j r_1^{t_j} = r_1^{1-n}\left\{-a^* D_j + \frac{E_j}{gH_o}\right\} \quad \text{. . . . . (17a)}$$

$$a_j r_2^{s_j} + b_j r_2^{t_j} = r_2^{1-n}\{-a^* D_j\} \quad \text{. . . . . (17b)}$$

which can be solved to yield

$$a_j = \frac{a^* D_j [t_j r_1^{t_j} r_2^{1-n} - r_2^{t_j} r_1^{1-n}] + \dfrac{r_1^{1-n} r_2^{t_j} E_j}{gH_o}}{s_j r_1^{s_j} r_2^{t_j} - t_j r_1^{t_j} r_2^{s_j}} \quad \text{(17c)}$$

$$b_j = \frac{-a^* D_j [s_j r_1^{s_j} r_2^{1-n} - r_2^{s_j} r_1^{1-n}] - \dfrac{r_1^{1-n} r_2^{s_j} E_j}{gH_o}}{s_j r_2^{s_j} r_2^{t_j} - t_j r_1^{t_j} r_2^{s_j}} \quad \text{(17d)}$$

Next, consider the case of a unit wind stress in the direction $\theta = \phi$. If we let $\alpha = \theta - \phi$, it is readily verified that the solution is again given by Eq. 15a, with α substituted for θ throughout. A wind stress of arbitrary magnitude and direction can be decomposed into components in the directions $\theta = 0$ and $\theta = \phi$:

$$\mathbf{W} = W_o \mathbf{r}_o + W_\phi \mathbf{r}_\phi \quad \text{(18)}$$

in which $\mathbf{r}_o$ and $\mathbf{r}_\phi$ are unit vectors in the directions $\theta = 0$ and $\theta = \phi$, and W_o and W_ϕ are the components of wind stress in these directions. Thus, the complete steady state wind setup is obtained by superposition:

$$\zeta_W(r,\theta) = a^* r^{1-n} \{W_o \cos [(1-n)^{1/2}\theta] + W_\phi \cos [(1-n)^{1/2}(\theta - \phi)]\}$$
$$+ \sum_{j=0}^{\infty} \left\{ a_j r^{s_j} + b_j r^{t_j} \right\} \left\{ W_o \cos \left(\frac{j\pi\theta}{\phi} \right) + W_\phi \cos \left[\frac{j\pi(\theta - \phi)}{\phi} \right] \right\} \quad \text{(19)}$$

Periodic Tidal Response

To solve Eqs. 9, assume a solution of the form $\zeta_f(r,\theta,t) = \mathrm{Re}\{R(r)T(\theta)e^{i\omega t}\}$. Substituting this in Eq. 9a produces

$$\frac{1}{R}\left[r^2 R'' + rR'(1+n) + \frac{\beta^2 R}{r^{n-2}} \right] = \kappa^2 \quad \text{(20a)}$$

and $$\frac{T''}{T} = -\kappa^2 \quad \text{(20b)}$$

in which κ^2 = a separation constant and $\beta^2 = (\omega^2 - i\omega\tau)/gH_o$. The general solution is thus

$$\zeta_f(r,\theta,t) = \mathrm{Re}\left\{ \sum_j (a_j R_{1j} + b_j R_{2j})[\cos(\kappa_j\theta) + c_j \sin(\kappa_j\theta)]\, e^{i\omega t} \right\} \quad \text{(21)}$$

in which $R_{1j}(r)$ and $R_{2j}(r)$ are the complex solutions of Eq. 20a. Boundary condition Eq. 9d is satisfied by letting $c_j = 0$ for all j, and by retaining only those terms for which $\kappa_j = j\pi/\phi$, j = 0, 1, 2 .... If boundary condition Eq. 9c is expressed as a Fourier series:

$$\zeta(\theta) = \sum_{j=0}^{\infty} F_j \cos\left(\frac{j\pi\theta}{\phi}\right) \quad \text{(22)}$$

in which $F_j = \int_0^\theta \zeta_o(\theta) \cos (j\pi\theta/\phi) d\theta / \int_0^\phi \cos^2(j\pi\theta/\phi) d\theta$, the remaining boundary conditions can be applied to determine the complex constants a_j and b_j:

$$a_j R'_{1j}(r_1) + b_j R'_{2j}(r_1) = 0 \quad \text{(23a)}$$

$$a_j R_{1j}(r_2) + b_j R_{2j}(r_2) = F_j \quad \text{(23b)}$$

which yields

$$a_j = \frac{F_j R'_{2j}(r_1)}{R'_{2j}(r_1)R_{1j}(r_2) - R_{2j}(r_2)R'_{1j}(r_1)} \quad \text{(23c)}$$

$$b_j = \frac{-F_j R'_{1j}(r_1)}{R'_{2j}(r_1)R_{1j}(r_2) - R_{2j}(r_2)R'_{1j}(r_1)} \quad \text{(23d)}$$

The complete solution is thus

$$\zeta_f(r,\theta,t) = \text{Re}\left\{e^{i\omega t}\sum_{j=0}^{\infty}(a_j R_{1j} + b_j R_{2j})\left[\cos\left(\frac{j\pi\theta}{\phi}\right)\right]\right\} \quad \text{(24)}$$

The functions R_{1j} and R_{2j} are given for any value of $n \neq 2$ by Hildebrand (6):

$$R_{1j}(r) = r^{-n/2} J_p\left[\frac{\beta}{1-\dfrac{n}{2}} r^{1-(n/2)}\right] \quad \text{(25a)}$$

$$R_{2j}(r) = r^{-n/2} Y_p\left[\frac{\beta}{1-\dfrac{n}{2}} r^{1-(n/2)}\right] \quad \text{(25b)}$$

in which J_p and Y_p are the solutions of Bessel's equation of order p, and

$$p = \frac{1}{2-n}\sqrt{n^2 + \left(\frac{2j\pi}{\phi}\right)^2} \quad \text{(25c)}$$

When $n = 2$ or $\beta = 0$, the limiting forms of Eqs. 25*a* and 25*b* are

$$R_{1j}(r) = r^{s_j};\quad s_j = -\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 - \beta^2 + \left(\frac{j\pi}{\phi}\right)^2} \quad \text{(25d)}$$

$$R_{2j}(r) = r^{t_j};\quad t_j = -\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 - \beta^2 + \left(\frac{j\pi}{\phi}\right)^2} \quad \text{(25e)}$$

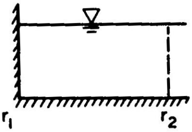
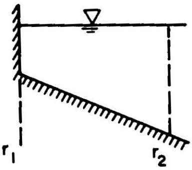
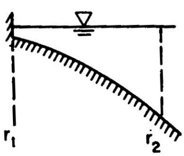
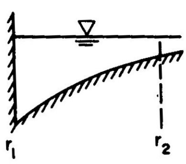
Since Eqs. 9 are linear, the response to forcing at several frequencies can be obtained separately and superimposed. Thus, to accomodate a generalized version of boundary condition Eq. 9*c* such as

$$\zeta_f(r_2,\theta,t) = \text{Re}\left\{\sum_{j=1}^{N}\zeta_j(\theta)e^{i\omega_j t}\right\}$$

it is only necessary to obtain N individual solutions as previously seen. Situations where a time-independent elevation $\zeta_j(\theta)$ is imposed at $r = r_2$ can be handled by letting $\omega = 0$ in Eq. 9c.

When ζ_o in Eq. 9c is independent of θ, the solution given by Eq. 24 is greatly simplified because a_j and b_j will be zero for $j \geq 1$. The response is thus independent of θ. Table 1 gives, for a few values of n, the complete

TABLE 1.—Some Periodic One-Dimensional Polar Solutions with Frequency ω

n (1)	Bathymetry (2)	Solution (3)	Constants (4)
0		$\zeta(r,t) = \mathrm{Re}\{[AJ_o(\beta r) + BY_o(\beta r)]e^{i\omega t}\}$ $V_r(r,t) = \mathrm{Re}\left\{[-AJ_1(\beta r) - BY_1(\beta r)]\frac{i\omega}{\beta H_o}e^{i\omega t}\right\}$	$A = \frac{\zeta_o Y_1(\beta r_1)}{J_o(\beta r_2)Y_1(\beta r_1) - J_1(\beta r_1)Y_o(\beta r_2)}$ $B = \frac{-\zeta_o J_1(\beta r_1)}{J_o(\beta r_2)Y_1(\beta r_1) - J_1(\beta r_1)Y_o(\beta r_2)}$
1		$\zeta(r,t) = \mathrm{Re}\{1/\sqrt{r}\,[AJ_1(2\beta\sqrt{r}) + BY_1(2\beta\sqrt{r})]e^{i\omega t}\}$ $V_r(r,t) = \mathrm{Re}\left\{1/r\,[-AJ_2(2\beta\sqrt{r}) - BY_2(2\beta\sqrt{r})]\frac{i\omega}{\beta H_o}e^{i\omega t}\right\}$	$A = \zeta_o\sqrt{r_2}\,Y_2(2\beta\sqrt{r_1})/[J_1(2\beta\sqrt{r_2})Y_2(2\beta\sqrt{r_1}) - J_2(2\beta\sqrt{r_1})Y_1(2\beta\sqrt{r_2})]$ $B = -\zeta_o\sqrt{r_2}\,J_2(2\beta\sqrt{r_1})/[J_1(2\beta\sqrt{r_2})Y_2(2\beta\sqrt{r_1}) - J_2(2\beta\sqrt{r_1})Y_1(2\beta\sqrt{r_2})]$
2		$\zeta(r,t) = \mathrm{Re}\{[Ar^{s_1} + Br^{s_2}]e^{i\omega t}\}$ $V_r(r,t) = \mathrm{Re}\left\{[s_1Ar^{s_1-1} + s_2Br^{s_2-1}]\frac{i\omega}{\beta^2 H_o}e^{i\omega t}\right\}$	$A = \frac{\zeta_o s_2 r_1^{s_2}}{[s_2 r_2^{s_1} r_1^{s_2} - s_1 r_1^{s_1} r_2^{s_2}]}$ $B = \frac{-\zeta_o s_1 r_1^{s_1}}{s_2 r_2^{s_1} r_1^{s_2} - s_1 r_1^{s_1} r_2^{s_2}}$ $s_1, s_2 = -1 \pm \sqrt{1-\beta^2}$
−2		$\zeta = \mathrm{Re}\left[\zeta_o\frac{\cosh\left[\frac{\beta}{2}(r^2 - r_1^2)\right]}{\cosh\left[\frac{\beta}{2}(r_2^2 - r_1^2)\right]}e^{i\omega t}\right]$ $v_r = \mathrm{Re}\left[\zeta_o r\frac{\sinh\left[\frac{\beta}{2}(r^2 - r_1^2)\right]}{\cosh\left[\frac{\beta}{2}(r_2^2 - r_1^2)\right]}\frac{i\omega}{\beta H_o}e^{i\omega t}\right]$	

Note: $h(r) = H_o r^n$; $\beta^2 = (\omega^2 - i\omega\tau)/gH_o$.

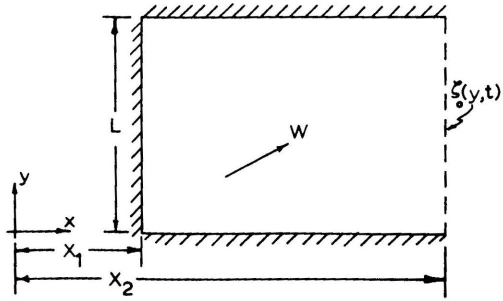


FIG. 2.—Rectangular Section in x-y Coordinates with Opening at $x = x_2$

solution to Eq. 9 in this special case. The solution for $n = 0$ in the absence of friction has been examined by Lamb (9).

Case II: Cartesian Geometry.—The Cartesian version of Case I is depicted in Fig. 2. Flow is tangent to the solid boundaries at $x = x_1$, $y = 0$, and $y = L$. A tidal forcing function is specified at $x = x_2$, and a constant wind stress is present everywhere. Bathymetry is described by $h = H_o x^n$, in which H_o is a constant, and n is not necessarily an integer and may assume any real value. Boundary conditions are:

$$\text{at} \quad x = x_1: \quad \frac{\partial \zeta}{\partial x} - \frac{W_x}{gh} = 0 \quad \text{(26a)}$$

$$\text{at} \quad x = x_2: \quad \zeta(x_2, y, t) = \text{Re}\,\{\zeta_o(y) e^{i\omega t}\} \quad \text{(26b)}$$

$$\text{at} \quad y = 0, L: \quad \frac{\partial \zeta}{\partial y} - \frac{W_y}{gh} = 0 \quad \text{(26c)}$$

in which W_x, W_y = the wind stress components in the x and y directions; ω = the frequency of the tidal forcing function; $\zeta_o(y)$ = a complex function representing the tidal amplitude and phase at $x = x_2$; and $i = \sqrt{-1}$. Solution of this problem has been examined by Briggs and Madsen (2) for the case of zero wind stress, zero friction, and constant bathymetry.

As in the polar case, the solution to Eq. 5 subject to Eq. 26 can be decomposed:

$$\zeta = \zeta_f + \zeta_W \quad \text{(27)}$$

in which ζ_f = the solution in the absence of wind, and ζ_W = the solution in the absence of tidal forcing. The two problems to be solved are thus

$$\frac{\partial^2 \zeta_W}{\partial x^2} + \frac{\partial^2 \zeta_W}{\partial y^2} + \frac{n}{x}\frac{\partial \zeta_W}{\partial x} = 0 \quad \text{(28a)}$$

$$\text{at} \quad x = x_1: \quad \frac{\partial \zeta_W}{\partial x} - \frac{W_x}{gH_o x^n} = 0 \quad \text{(28b)}$$

$$\text{at} \quad x = x_2: \quad \zeta_W(x_2, y) = 0 \quad \text{(28c)}$$

$$\text{at} \quad y = 0, L: \quad \frac{\partial \zeta_W}{\partial y} - \frac{W_y}{gH_o x^n} = 0 \quad \text{(28d)}$$

and

$$\frac{\partial^2 \zeta_f}{\partial t^2} + \tau \frac{\partial \zeta_f}{\partial t} - gH_o x^n \left[\frac{\partial^2 \zeta_f}{\partial x^2} + \frac{\partial^2 \zeta_f}{\partial y^2} + \frac{n}{x}\frac{\partial \zeta_f}{\partial x}\right] = 0. \quad \text{(29}$$

$$\text{at} \quad x = x_1: \quad \frac{\partial \zeta_f}{\partial x} = 0 \quad \text{(29b)}$$

$$\text{at} \quad x = x_2: \quad \zeta_f(x_2, y, t) = \text{Re}\,\{\zeta_o(y) e^{i\omega t}\} \quad \text{(29c)}$$

$$\text{at} \quad y = 0, L: \quad \frac{\partial \zeta_f}{\partial y} = 0 \quad \text{(29d)}$$

Steady State Wind Setup

Eqs. 28 can be solved in a manner similar to that used for Eqs. 8. First, consider a unit wind stress in the x direction. Assuming a solution of the form $\zeta_W(x,y) = \xi(x)\,\eta(y)$ in Eq. 28a results in

$$\frac{1}{\xi}\left[\xi'' + \frac{n}{x}\xi'\right] = \kappa^2 \quad \text{(30a)}$$

and $$\frac{\eta''}{\eta} = -\kappa^2 \quad \text{(30b)}$$

in which κ^2 = a separation constant. The general solution to Eq. 30 is

$$\zeta_W(x,y) = \sum_j x^p \{a_j J_p(i\kappa_j x) + b_j Y_p(i\kappa_j x)\}\{\cos(\kappa_j y) + c_j \sin(\kappa_j y)\} \quad \text{(31)}$$

$$p = \frac{1-n}{2} \quad \text{(32)}$$

in which J_p and Y_p are Bessel functions of order p, and $\sin(\kappa_j y)$ is replaced by y when $\kappa_j = 0$. Boundary conditions are:

$$\text{at } x = x_1:\quad \frac{\partial \zeta_W}{\partial x} = \frac{1}{gH_o x_1^n} \quad \text{(33a)}$$

$$\text{at } x = x_2:\quad \zeta_W = 0 \quad \text{(33b)}$$

$$\text{at } y = 0, L:\quad \frac{\partial \zeta_W}{\partial y} = 0 \quad \text{(33c)}$$

Eq. 33c is satisfied by letting $c_j = 0$ for all j, and by retaining only those terms for which $\kappa_j = j\pi/L$, j = 0, 1, 2, ... Since boundary conditions Eqs. 33a and 33b do not depend on y, they are satisfied by retaining only the term with $j = 0$. The solution is thus the limiting form of Eq. 31 when $\kappa_j = 0$:

$$\zeta_W(x,y) = a + bx^{1-n} \quad (n \neq 1) \quad \text{(34a)}$$

$$\zeta_W(x,y) = a + b\log x \quad (n = 1) \quad \text{(34b)}$$

The constants a and b are determined by Eqs. 33a and 33b:

$$a = -\frac{x_2^{1-n}}{(1-n)gH_o};\quad b = \frac{1}{(1-n)gH_o} \quad (n \neq 1) \quad \text{(35a)}$$

$$a = -\frac{\log x_2}{gH_o};\quad b = \frac{1}{gH_o} \quad (n = 1) \quad \text{(35b)}$$

Next, consider the case of a unit wind stress in the y direction. The general solution is again given by Eqs. 31, 32, which is rewritten for convenience as:

$$\ldots y) = \sum_j x^p \{J_p(i\kappa_j x) + b_j Y_p(i\kappa_j x)\}\{c_j \cos(\kappa_j y) + d_j \sin(\kappa_j y)\} \quad \text{(36)}$$

$$p = \frac{1 - n}{2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (37)$$

Boundary conditions are:

$$\text{at} \quad x = x_1: \quad \frac{\partial \zeta_W}{\partial x} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (38a)$$

$$\text{at} \quad x = x_2: \quad \zeta_W = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (38b)$$

$$\text{at} \quad y = 0, L: \quad \frac{\partial \zeta_W}{\partial y} = \frac{1}{gH_o x^n} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (38c)$$

Eq. 38*b* is satisfied by proper choice of b_j:

$$b_j = -\frac{J_p(i\,\kappa_j x_2)}{Y_p(i\,\kappa_j x_2)} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (39)$$

If one makes the substitution

$$Z_p(i\,\kappa_j x) = J_p(i\kappa_j x)\, Y_p(i\kappa_j x_2) - J_p(i\kappa_j x_2)\, Y_p(i\kappa_j x)$$

then Eq. 38*a* is satisfied by selecting only those values of κ_j for which

$$\frac{d}{dx}\,[x^p Z_p(i\,\kappa_j x)]\,\big|_{x=x_1} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (40)$$

Expanding this produces

$$Z_{p-1}(i\,\kappa_j x_1) = J_{p-1}(i\,\kappa_j x_1)\, Y_p(i\,\kappa_j x_2) - J_p(i\,\kappa_j x_2)\, Y_{p-1}(i\,\kappa_j x_1) = 0 \quad \ldots (41)$$

Thus, the quantity $(i\,\kappa_j x_1)$ may only assume the values of the zeros of the function Z_{p-1}. It is clear from this that κ_j is limited to a discrete set of imaginary numbers:

$$\kappa_j = -\frac{i}{x_1}\gamma_j \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (42)$$

in which γ_j, the jth zero of Z_{p-1}, is real. Eq. 36 now becomes

$$\zeta_W(x, y) = x^p \sum_{j=1}^{\infty} \left\{ Z_p\left(\frac{\gamma_j x}{x_1}\right)\right\}\left\{ f_j \cosh\left(\frac{\gamma_j y}{x_1}\right) + e_j \sinh\left(\frac{\gamma_j y}{x_1}\right)\right\} \quad \ldots (43)$$

To satisfy Eq. 38*c*, first expand the function $x^{-(n+p)}$ in terms of the complete, orthogonal set of functions Z_p:

$$x^{-(n+p)} = \sum_{j=1}^{\infty} G_j Z_p\left(\frac{\gamma_j}{x_1}x\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (44a)$$

$$\text{in which} \quad G_j = \frac{\displaystyle\int_{x_1}^{x_2} x^p Z_p\left(\frac{\gamma_j}{x_1}x\right) dx}{\displaystyle\int_{x_1}^{x_2} x\left[Z_p\left(\frac{\gamma_j}{x_1}x\right)\right]^2 dx} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (44b)$$

The numerator and denominator of this expression are easily evaluated using formulas in Luke (11) and Wylie (15), respectively. Application of boundary condition Eq. 38c now gives a determination of the constants f_j and e_j:

$$\frac{\gamma_j}{x_1} e_j = \frac{G_j}{gH_o} \quad \text{(45}a\text{)}$$

$$\left[\frac{\gamma_j}{x_1}\sinh\left(\frac{\gamma_j}{x_1}L\right)\right] f_j + \left[\frac{\gamma_j}{x_1}\cosh\left(\frac{\gamma_j}{x_1}L\right)\right] e_j = \frac{G_j}{gH_o} \quad \text{(45}b\text{)}$$

which produces $$f_j = \frac{x_1 G_j}{\gamma_j g H_o}\left[\frac{1 - \cosh\left(\dfrac{\gamma_j L}{x_1}\right)}{\sinh\left(\dfrac{\gamma_j L}{x_1}\right)}\right] \quad \text{(45}c\text{)}$$

$$e_j = \frac{x_1 G_j}{\gamma_j g H_o} \quad \text{(45}d\text{)}$$

The complete steady-state response to an arbitrary wind stress is thus obtained by superposition:

$$\zeta_W(x,y) = W_x\{a + bx^{1-n}\} + W_y\left\{x^p \sum_{j=1}^{\infty} Z_p\left(\frac{\gamma_j x}{x_1}\right)\left[f_j \cosh\left(\frac{\gamma_j y}{x_1}\right) + e_j \sinh\left(\frac{\gamma_j y}{x_1}\right)\right]\right\} \quad \text{(46)}$$

in which W_x and W_y are the x and y components of the wind stress, respectively.

Periodic Tidal Response

Following the approach taken in the polar case, Eqs. 29 can be solved by assuming a solution of the form $\zeta_f(x,y,t) = \text{Re}\{\xi(x)\,\eta(y)e^{i\omega t}\}$. Substituting this into Eq. 29a results in

$$\frac{1}{\xi}\left[\xi'' + \frac{n}{x}\xi' + \frac{\beta^2}{x^n}\xi\right] = \kappa^2 \quad \text{(47}a\text{)}$$

$$\frac{\eta''}{\eta} = -\kappa^2 \quad \text{(47}b\text{)}$$

in which κ^2 is a separation constant and $\beta^2 = (\omega^2 - i\omega\tau)/gH_o$ as previously seen. The general solution is thus

$$\zeta_f(x,y,t) = \text{Re}\left\{\sum_{j=0}^{\infty} (a_j\xi_{1j} + b_j\xi_{2j})[\cos(\kappa_j y) + c_j \sin(\kappa_j y)]\, e^{i\omega t}\right\} \quad \text{(48)}$$

in which $\xi_{1j}(x)$ and $\xi_{2j}(x)$ are the complex solutions of Eq. 47a. Boundary condition Eq. 29d is satisfied by letting $c_j = 0$ for all j, and by retaining only those terms in Eq. 48 for which $\kappa_j = j\pi/L$, $j = 0, 1, 2, \ldots$ Boundary condition Eq. 29c can be expressed as a Fourier series:

$$\zeta_o(y) = \sum_{j=0}^{\infty} H_j \cos\left(\frac{j\pi y}{L}\right) \quad \text{(49)}$$

in which $H_j = \int_0^L \zeta_o \cos(j\pi y/L)\, dy / \int_0^L \cos^2(j\pi y/L)\, dy$. The remaining conditions Eqs. 29b and 29c can now be applied to determine the complex constants a_j and b_j, which yield expressions identical in form to Eq. 23:

$$a_j \xi'_{1j}(x_1) + b_j \xi'_{2j}(x_1) = 0 \quad \text{(50a)}$$

$$a_j \xi_{1j}(x_2) + b_j \xi_{2j}(x_2) = H_j \quad \text{(50b)}$$

$$a_j = \frac{H_j \xi'_{2j}(x_1)}{\xi'_{2j}(x_1)\xi_{1j}(x_2) - \xi_{2j}(x_2)\xi'_{1j}(x_1)} \quad \text{(50c)}$$

$$b_j = \frac{-H_j \xi'_{1j}(x_1)}{\xi'_{2j}(x_1)\xi_{1j}(x_2) - \xi_{2j}(x_2)\xi'_{1j}(x_1)} \quad \text{(50d)}$$

The complete solution is thus

$$\zeta_f(x, y, t) = \mathrm{Re}\left\{ e^{i\omega t} \sum_{j=0}^{\infty} (a_j \xi_{1j} + b_j \xi_{2j}) \left[\cos\left(\frac{j\pi y}{L}\right) \right] \right\} \quad \text{(51)}$$

The functions $\xi_{1j}(x)$ and $\xi_{2j}(x)$ have been obtained for a few specific values of n:

$$n = 0: \quad \xi_{1j}(x) = \cosh\left\{ \left[\left(\frac{j\pi}{L}\right)^2 - \beta^2 \right] x \right\} \quad \text{(52a)}$$

$$\xi_{2j}(x) = \sinh\left\{ \left[\left(\frac{j\pi}{L}\right)^2 - \beta^2 \right] x \right\} \quad \text{(52b)}$$

$$n = 1: \quad \xi_{1j}(x) = e^{j\pi x/L} \left[M\left(\frac{1}{2} + \frac{\beta^2 L}{2j\pi};\ 1;\ -\frac{2j\pi x}{L} \right) \right] \quad \text{(52c)}$$

$$\xi_{2j}(x) = e^{j\pi x/L} \left[U\left(\frac{1}{2} + \frac{\beta^2 L}{2j\pi} \right);\ 1;\ -\frac{2j\pi x}{L} \right) \Big] \quad \text{(52d)}$$

$$n = 2: \quad \xi_{1j}(x) = x^{-(1/2)} J_p\left(\frac{ij\pi x}{L} \right) \quad \text{(52e)}$$

$$\xi_{2j}(x) = x^{-(1/2)} Y_p\left(\frac{ij\pi x}{L} \right); \quad p = \sqrt{\frac{1}{4} - \beta^2} \quad \text{(52f)}$$

$$n = -2: \quad \xi_{1j}(x) = e^{-(1/2)i\beta x^2} \left[M\left(\frac{5}{4} - i\frac{j^2\pi^2}{4L^2};\ \frac{5}{2};\ i\beta x^2 \right) \right] \quad \text{(52g)}$$

$$\xi_{2j}(x) = e^{-(1/2)i\beta x^2} \left[U\left(\frac{5}{4} - i\frac{j^2\pi^2}{4L^2};\ \frac{5}{2};\ i\beta x^2 \right) \right] \quad \text{(52h)}$$

in which J_p, Y_p = the solutions of Bessel's equation of order p, and M, U = the solutions of Kummer's equation (13).

As in the polar case, the response to forcing at several frequencies, including $\omega = 0$, may be superimposed to obtain the solution to a more general version of boundary condition Eq. 29c. Again, when ζ_o in Eq. 29c is independent of y, the solution contains only one component ($j = 0$) of the series Eq.

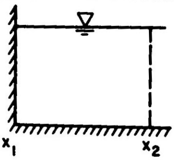
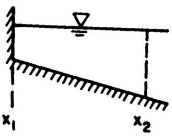
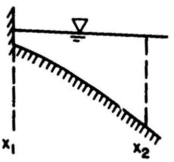
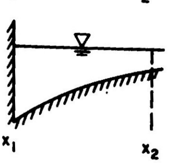
and is independent of y. The solution for this special case *for any value of* $n \neq 2$ is obtained by solving Eq. 47*a* with $\kappa^2 = 0$:

$$\xi_1(x) = x^{(1-n)/2} J_p\left[\frac{2\beta x^{1-(n/2)}}{2-n}\right] \quad \text{(53a)}$$

$$\xi_2(x) = x^{(1-n)/2} Y_p\left[\frac{2\beta x^{1-(n/2)}}{2-n}\right] \quad \text{(53b)}$$

$$p = \frac{1-n}{2-n} \quad \text{(53c)}$$

TABLE 2.—Some Periodic One-Dimensional Cartesian Solutions with Frequency ω

n (1)	Bathymetry (2)	Solution (3)	Constants (4)
0		$\zeta = \mathrm{Re}\left\{\zeta_o e^{i\omega t}\frac{\cos[\beta(x-x_1)]}{\cos[\beta(x_2-x_1)]}\right\}$ $V_x = \mathrm{Re}\left\{-\frac{i\omega\zeta_o}{\beta H_o} e^{i\omega t}\frac{\sin[\beta(x-x_1)]}{\cos[\beta(x_2-x_1)]}\right\}$	
1		$\zeta(x,t) = \mathrm{Re}\{[AJ_o(2\beta\sqrt{x}) + BY_o(2\beta\sqrt{x})]e^{i\omega t}\}$ $V_x(x,t) = \mathrm{Re}\left\{\frac{1}{\sqrt{x}}[-AJ_1(2\beta\sqrt{x}) - BY_1(2\beta\sqrt{x})]\frac{i\omega}{\beta H_o}e^{i\omega t}\right\}$	$A = \zeta_o Y_1(2\beta\sqrt{x_1})/[J_o(2\beta\sqrt{x_2})Y_1(2\beta\sqrt{x_1}) - Y_o(2\beta\sqrt{x_2})J_1(2\beta\sqrt{x_1})]$ $B = -\zeta_o J_1(2\beta\sqrt{x_1})/[J_o(2\beta\sqrt{x_2})Y_1(2\beta\sqrt{x_1}) - Y_o(2\beta\sqrt{x_2})J_1(2\beta\sqrt{x_1})]$
2		$\zeta(x,t) = \mathrm{Re}\{[A x^{s_1} + B x^{s_2}]e^{i\omega t}\}$ $V_x(x,t) = \mathrm{Re}\left\{[As_1 x^{s_1-1} + Bs_2 x^{s_2-1}]\frac{i\omega}{\beta^2 H_o}e^{i\omega t}\right\}$	$A = \frac{\zeta_o s_2 x_1^{s_2}}{s_2 x_1^{s_2} x_2^{s_1} - s_1 x_1^{s_1} x_2^{s_2}}$ $B = \frac{-\zeta_o s_1 x_1^{s_1}}{s_2 x_1^{s_2} x_2^{s_1} - s_1 x_1^{s_1} x_2^{s_2}}$ $s_1, s_2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \beta^2}$
−2		$\zeta(x,t) = \mathrm{Re}\left\{x^{3/2}\left[AJ_{3/4}\left(\frac{\beta x^2}{2}\right) + BY_{3/4}\left(\frac{\beta x^2}{2}\right)\right]e^{i\omega t}\right\}$ $V_x(x,t) = \mathrm{Re}\left\{x^{5/2}\left[AJ_{-(1/4)}\left(\frac{\beta x^2}{2}\right) + BY_{-(1/4)}\left(\frac{\beta x^2}{2}\right)\right]\frac{i\omega}{\beta H_o}e^{i\omega t}\right\}$	$A = \zeta_o Y_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right) \Big/ \left[J_{3/4}\left(\frac{\beta x_2^2}{2}\right) Y_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right) - Y_{3/4}\left(\frac{\beta x_2^2}{2}\right) J_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right)\right]$ $B = -\zeta_o J_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right) \Big/ \left[J_{3/4}\left(\frac{\beta x_2^2}{2}\right) Y_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right) - Y_{3/4}\left(\frac{\beta x_2^2}{2}\right) J_{-(1/4)}\left(\frac{\beta x_1^2}{2}\right)\right]$

Note: $h(x) = H_o x^n$; $\beta^2 = (\omega^2 - i\omega\tau)/gH_o$.

in which J_p and Y_p are the Bessel functions as previously seen. When $n = 2$, the limiting form is:

$$\xi_1(x) = x^s; \quad s = -\frac{1}{2} + \sqrt{\frac{1}{4} - \beta^2} \quad \text{(53d)}$$

$$\xi_2(x) = x^t; \quad t = -\frac{1}{2} - \sqrt{\frac{1}{4} - \beta^2} \quad \text{(53e)}$$

Table 2 gives, for certain values of n, the complete solution to Eq. 29 when ζ_o is constant. The solutions for $n = 0$ and 1, in the absence of friction, have been examined by Lamb (9). Ippen (7) has examined the case $n = 0$ with linearized frictional dissipation.

Transient Solutions

Superposition in Frequency Domain.—An interesting example of the use of superposition to obtain boundary forcing functions composed of various temporal frequencies is the following approach to the "cold start" problem. Suppose boundary condition Eq. 6*b* is represented by

$$\zeta(r_2,\theta,t) = 0; \quad t < \tilde{T}(\theta) \quad \text{. . . . . (54a)}$$

$$\zeta(r_2,\theta,t) = \text{Re}\{\tilde{F}(\theta)[-ie^{i\omega(t-\tilde{T}(\theta))}]\}; \quad t \geq \tilde{T}(\theta) \quad \text{. . . . . (54b)}$$

in which the real functions $\tilde{F}(\theta)$ and $\tilde{T}(\theta)$ are the amplitude and phase,

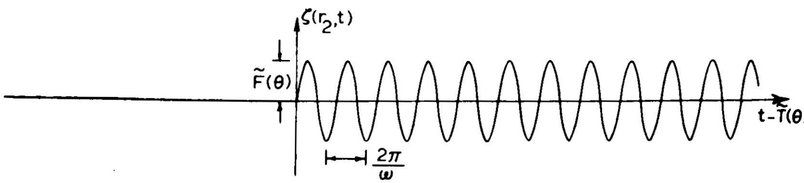


FIG. 3.—Boundary Condition for Cold Start Problem

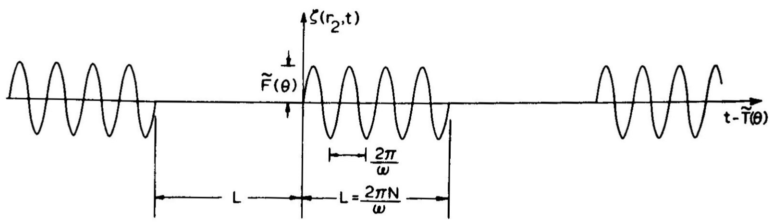


FIG. 4.—Approximate Boundary Condition for Cold Start Problem

respectively, of the sinusoidal forcing function at $r = r_2$. For a given value of θ, Eq. 54 can be plotted as in Fig. 3. As an approximation, replace the boundary condition with the periodic function illustrated in Fig. 4, and consider the dynamic steady state response of the system, which will be periodic with period $2L$. If N is sufficiently large to allow the system to alternately come to rest and reach the dynamic steady state with frequency ω, then this problem will approximate the "cold start" problem. (Appropriate values for N can be determined by examination of the fully transient problem, subsequently shown.) The function $\zeta_o(r_2,\theta,t)$ in this case is given by the Fourier series

$$\zeta_o(r_2,\theta,t) = \text{Re}\left\{\tilde{F}(\theta)\sum_{j=1}^{\infty} a_j e^{i\omega_j[t-\tilde{T}(\theta)]} + \frac{\tilde{F}(\theta)}{2}(-ie^{i\omega[t-\tilde{T}(\theta)]})\right\} \quad \text{. . . . (55a)}$$

in which $$\omega_j = \frac{(2j-1)\omega}{2N} \quad \text{(55}b\text{)}$$

and $$a_j = \frac{1}{\pi}\left(\frac{N}{N^2 - j^2 + j - \frac{1}{4}}\right) \quad \text{(55}c\text{)}$$

The response can thus be obtained by a straightforward application of superposition. The Cartesian case can be treated by the same procedure.

Full Solutions for Cold Start Problem.—By using the periodic steady state solutions examined previously, it is possible to obtain fully transient solutions to the linearized long wave equations. By way of example, the case of startup from rest in the polar sector will be considered. Let the full solution for ζ be described by

$$\zeta = \zeta_{DS}(r,\theta,t) + \chi(r,\theta,t)$$

in which ζ_{DS} = the periodic solution previously obtained. Then, with $h = H_o r^n$, the equation for χ with appropriate boundary conditions which must be solved is

$$\frac{1}{gH_o}\left[\frac{\partial^2\chi}{\partial t^2} + \tau\frac{\partial\chi}{\partial t}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left(r^{n+1}\frac{\partial\chi}{\partial r}\right) - r^{n-2}\frac{\partial^2\chi}{\partial\theta^2} = 0 \quad \text{(56}a\text{)}$$

at $$r = r_1,\ \frac{\partial\chi}{\partial r} = 0 \quad \text{(56}b\text{)}$$

at $$r = r_2,\ \chi = 0 \quad \text{(56}c\text{)}$$

at $$\theta = 0,\ \phi,\ \frac{\partial\chi}{\partial\theta} = 0 \quad \text{(56}d\text{)}$$

at $$t = 0 \quad \chi = -\zeta_{DS}(r,\theta,0) \quad \text{(56}e\text{)}$$

at $$t = 0 \quad \frac{\partial\chi}{\partial t} = -\frac{\partial\zeta_{DS}}{\partial t}(r,\theta,0) \quad \text{(56}f\text{)}$$

Assuming a solution of the form $\chi(r,\theta,t) = R(r)T(\theta)\,e^{i\omega t}$ where

$$\omega = \frac{i\tau}{2} \pm \sqrt{\lambda^2 gH_o - \left(\frac{\tau}{2}\right)^2} \quad \text{(57}a\text{)}$$

one finds that the spatial part of χ must satisfy

$$\frac{1}{R}\left[r^2R'' + (n+1)rR' + \frac{\lambda^2 R}{r^{n-2}}\right] = \kappa^2 \quad \text{(57}b\text{)}$$

and $$\frac{T''}{T} = -\kappa^2 \quad \text{(57}c\text{)}$$

which are identical to Eqs. 20. The general solutions obtained previously can thus be used here. However, in Eqs. 20, the value of β^2 is dictated by the frequency of the forcing function; the corresponding term λ^2 in Eq. 57 represents free vibrations and in general will take on a spectrum of discrete values which are dictated by the homogeneous boundary conditions. Application of Eq. 57 gives the solution for $n \neq 2$:

$$\chi = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (A_{j,k} e^{i\omega_{+j,k}t} + B_{j,k} e^{i\omega_{-j,k}t}) \cos\left(\frac{j\pi\theta}{\phi}\right) r^{-(n/2)} R_{j,k}(r) \quad \ldots \quad (58a)$$

in which $$R_{j,k}(r) = \left[J_{\nu_j}\left(\frac{\lambda_{j,k}}{p} r_2^p\right) Y_{\nu_j}\left(\frac{\lambda_{j,k}}{p} r^p\right) - Y_{\nu_j}\left(\frac{\lambda_{j,k}}{p} r_2^p\right) J_{\nu_j}\left(\frac{\lambda_{j,k}}{p} r^p\right) \right] \quad \ldots \quad (58b)$$

$$\omega_{\pm j,k} = \frac{i\tau}{2} \pm \sqrt{-\left(\frac{\tau}{2}\right)^2 + \lambda_{j,k}^2 g H_o} \quad \ldots \quad (58c)$$

$$\nu_j = \frac{\sqrt{n^2 + \left(\frac{2j\pi}{\phi}\right)^2}}{(2-n)} \quad \ldots \quad (58d)$$

$$p = \frac{2-n}{2}, \text{ and} \quad \ldots \quad (58e)$$

$$\lambda_{j,k} \text{ are such that } \frac{d}{dr}[r^{-(n/2)} R_{j,k}(r)] \,|_{r_1} = 0 \quad \ldots \quad (58f)$$

Values of the complex constants $A_{j,k}$ and $B_{j,k}$ may be obtained from the initial conditions by exploiting the orthogonal properties of the spatial functions

$$A_{j,k} + B_{j,k} = \frac{-\int_0^{\phi}\int_{r_1}^{r_2} \zeta_{DS}(r,\theta,0) \cos\left[\frac{j\pi\theta}{\phi}\right] r^{1-(n/2)} R_{j,k}(r)\, dr d\theta}{\int_0^{\phi}\int_{r_1}^{r_2} r_2^{-(n/2)} \cos^2\left[\frac{j\pi\theta}{\phi}\right] r^{1-n} R_{j,k}^2(r)\, dr d\theta} \quad (58g)$$

$$i\omega_{+j,k} A_{j,k} + i\omega_{-j,k} B_{j,k} = -\frac{\int_0^{\phi}\int_{r_1}^{r_2} \frac{\partial \zeta_{DS}}{\partial t}(r,\theta,0) \cos\left[\frac{j\pi\theta}{\phi}\right] r^{1-(n/2)} R_{j,k}(r)\, dr d\theta}{\int_0^{\phi}\int_{r_1}^{r_2} r_2^{-(n/2)} \cos^2\left[\frac{j\pi\theta}{\phi}\right] r^{1-n} R_{j,k}^2(r)\, dr d\theta} \quad \ldots \quad (58h$$

When $n = 2$ the solution is somewhat simpler and given by

$$\chi = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \{A_{j,k} e^{i\omega_{+j,k}t} + B_{j,k} e^{i\omega_{-j,k}t} \quad \cos\left(\frac{j\pi\theta}{\phi}\right) R_{j,k}(r) \quad \ldots \quad (59a)$$

in which

$$R_{j,k}(r) = r_2^{\alpha_{+j,k}} r^{\alpha_{-j,k}} - r^{\alpha_{+j,k}} r_2^{\alpha_{-j,k}} \quad \ldots \quad (59b)$$

$$\omega_{\pm j,k} = \frac{i\tau}{2} \pm \sqrt{-\left(\frac{\tau}{2}\right)^2 + \lambda_{j,k}^2 g H_o} \quad \text{(59c)}$$

$$\alpha_{\pm j,k} = -1 \pm \sqrt{1 + \left(\frac{j\pi}{\phi}\right)^2 - \lambda_{j,k}^2} \quad \text{(59d)}$$

$\lambda_{j,k}^2$ are such that $\left.\dfrac{d}{dr} R_{j,k}(r)\right|_{r_1} = 0$

$$\text{or} \quad \frac{\alpha_{+j,k}}{\alpha_{-j,k}} = \left(\frac{r_2}{r_1}\right)^{\alpha_{+j,k} - \alpha_{-j,k}} \quad \text{(59e)}$$

The complex constants $A_{j,k}$ and $B_{j,k}$ are obtained from the orthogonality of the solution functions

$$A_{j,k} + B_{j,k} = \frac{\displaystyle\int_0^{\phi}\int_{r_1}^{r_2} \zeta_{DS}(r,\theta,0) \cos\left[\frac{j\pi\theta}{\phi}\right] r R_{j,k}(r)\, dr\, d\theta}{\displaystyle\int_0^{\phi}\int_{r_1}^{r_2} r \cos^2\left[\frac{j\pi\theta}{\phi}\right] R_{j,k}^2(r)\, dr\, d\theta} \quad \text{(59f)}$$

$$i\omega_{+j,k} A_{j,k} + i\omega_{-j,k} B_{j,k} = \frac{\displaystyle\int_0^{\phi}\int_{r_1}^{r_2} \frac{\partial \zeta_{DS}}{\partial t}(r,\theta,0) \cos\left[\frac{j\pi\theta}{\phi}\right] r R_{j,k}(r)\, dr\, d\theta}{\displaystyle\int_0^{\phi}\int_{r_1}^{r_2} r \cos^2\left[\frac{j\pi\theta}{\phi}\right] R_{j,k}^2(r)\, dr\, d\theta} \quad \text{(59g)}$$

An examination of the time dependence in Eqs. 58 and 59 reveals some interesting features. All of the transient solutions behave in time as

$$\chi(t) \sim e^{i\omega_{j,k} t} = e^{-\tau t/2 \pm it\sqrt{\lambda_{j,k}^2 g H_o - (\tau/2)^2}}$$

in which $\lambda_{j,k}^2$ is always real. For "large" values of $\lambda_{j,k}^2$, such that $\lambda_{j,k}^2 g H_o > (\tau/2)^2$, the solution will exhibit oscillatory behavior in time, within an envelope which decays exponentially at the rate $\tau/2$. Further, *all* oscillatory transients will decay at this same rate, so that when time after startup reaches the value $14/\tau$, all oscillatory transients will have decayed to approx 1% of their initial values.

For "small" or negative values of $\lambda_{j,k}^2$, such that $\lambda_{j,k}^2 g H \le (\tau/2)^2$, the solution will not oscillate in time:

$$\chi(t) \sim e^{-\tau t/2 \pm t\sqrt{(\tau/2)^2 - \lambda_{j,k}^2 g H_o}} \quad \text{(60)}$$

For non-negative values of $\lambda_{j,k}^2$, the transient solution will decay at a rate S, in which $0 \le S \le \tau$, with equality holding for the case $\lambda_{j,k}^2 = 0$. The condition $\lambda_{j,k}^2 < 0$ will result in an exponential growth of the transient solution. However, the boundary condition Eq. 58*f* makes solutions for $\lambda_{j,k}^2 \le 0$ inadmissable, the net result being that under the influence of friction, all transients decay, as expected.

Review

The solutions presented previously incorporate several features which should be of interest to those working with numerical models. Perhaps most notable is the inclusion of friction. Most models that do not introduce significant amounts of "numerical" damping will simply not overcome the effects of startup in a reasonable amount of time without the inclusion of a finite amount of friction. Yet analytic solutions with friction have been scarce, and limited to the simplest cases (Ippen, 7). The obstacles to most such solutions are overcome simply by the use of complex instead of real functions, which require little extra effort either in analysis or in computation.

The inclusion of wind stress, bathymetry which obeys a general power law, and two-dimensional circulation gives a broad spectrum of conditions against which model features can be tested. The availability of complete solutions for velocity in addition to surface elevation provides the means for verification of computed flow fields, which are in many practical cases the most important aspect of a problem.

The polar geometry solutions are especially interesting, since they incorporate some boundaries which are not straight. The importance of boundary geometry cannot be denied, and the accuracy with which a model depicts the dynamics at the boundary is certainly an important issue. Yet most comparisons of model

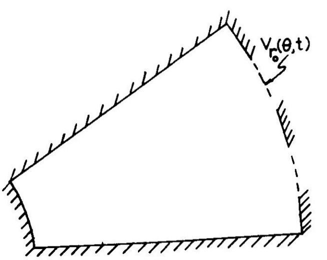


FIG. 5.—Annular Section in Polar Coordinates with Modified Opening

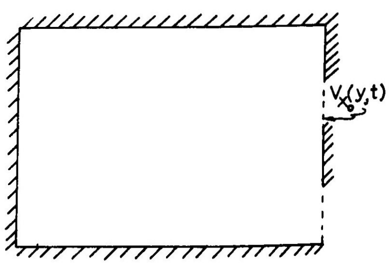


FIG. 6.—Rectangular Section in Cartesian Coordinates with Modified Opening

results with Cartesian solutions effectively minimize boundary error a priori, by aligning the boundaries with the grid directions. This advantage can seldom be realized in a real-world problem.

With slight modification, the solutions presented here can be applied to the situations depicted in Figs. 5 and 6, where at the external boundary $\mathbf{V}\cdot\mathbf{n}$ is specified as the tidal forcing function, in place of ζ. (Briggs and Madsen (2) have examined this problem for the frictionless, constant depth case in Cartesian geometry.) The only departure from the solutions presented herein is the change of boundary conditions Eqs. 6*b* and 26*b* such that the proper component of $\nabla\zeta$ is specified at the boundary, instead of ζ itself.

Although the evaluation of a two-dimensional solution involves in general an infinite series, considerable economy can be realized by choosing a tidal forcing function that requires only one or two Fourier components in space. This should in no way compromise the value of the solution for model verification purposes. The one-dimensional solutions provide an even simpler means of m verification when the effects of friction, bathymetry or wind, or all three of interest. The one-dimensional polar solutions give the added benefit o solution that is one-dimensional for analytic purposes but which is two-dimensi with curvilinear boundaries for models that use Cartesian coordinates.

Appendix.—References

1. Blumberg, A. F., "Numerical Model of Estuarine Circulation," *Journal of Hydraulics Division,* ASCE, Vol. 103, No. HY3, Proc. Paper 12815, Mar., 19 pp. 295–310.
2. Briggs, D. A., and Madsen, O. S., "Analytical Models for One- and Two-La Systems in Rectangular Basins," Part II of Mathematical Models of Massachuse Bay, Parsons Lab, Massachusetts Institute of Technology, Cambridge, Mass., *Rep No. 172,* 1973.
3. Cooley, R. L., and Moin, S. A., "Finite Element Solutions of Saint-Venant Equations *Journal of the Hydraulics Division,* ASCE, Vol. 102, No. HY6, Proc. Paper 1221 June, 1976, pp. 759–773.
4. Fread, D. L., "Implicit Dynamic Routing of Floods and Surges in the Low Mississippi," presented at the April 8–12, 1974, American Geophysical Union Sprin National Meeting, held at Washington, D.C.
5. Gallagher, R. H., Liggett, J. A., and Chan, S. T. K., "Finite Element Shallow Lak Circulation Analysis," *Journal of the Hydraulics Division,* ASCE, Vol. 99, No. HY Proc. Paper 9855, July, 1973, pp. 1083–1096.
6. Hildebrand, F. B., *Advanced Calculus for Applications,* Prentice-Hall, Englewoo Cliffs, N.J., 1962.
7. Ippen, A. T., "Tidal Dynamics in Estuaries, Part I: Estuaries of Rectangular Section," Chapter 10 of *Estuary and Coastline Hydrodynamics,* A. T. Ippen, ed., McGraw-Hi Book Co., Inc., New York, N.Y., 1966.
8. Kwizak, M., and Robert, A. J., "A Semi-Implicit Scheme for Grid Point Atmospheri Models of the Primitive Equations," *Monthly Weather Review,* Vol. 99, No. 1, Jan. 1971, pp. 32–36.
9. Lamb, H., *Hydrodynamics,* 6th ed., Dover Publications, New York, N.Y., 1945.
10. Leendertse, J. J., "Aspects of a Computational Model for Long-Period Water-Wave Propagation," *Rand Memorandum RM-5294-PR,* Santa Monica, Calif., 1967, pp. 165.
11. Luke, Y. L., "Integrals of Bessel Functions," *Handbook of Mathematical Functions,* M. Abramowitz and I. A. Stegun, eds., *National Bureau of Standards Applied Mathematics Series No. 55,* 1966, pp. 479–494.
12. Reid, R. O. and Bodine, B. R., "Numerical Model for Storm Surges in Galveston Bay," *Journal of the Waterways and Harbors Division,* ASCE, Vol. 94, No. WW1, Proc. Paper 5805, Feb., 1968, pp. 33–57.
13. Slater, L. J., "Confluent Hypergeometric Functions," *Handbook of Mathematical Functions,* M. Abramowitz and I. A. Stegun, eds., *National Bureau of Standards Applied Mathematics Series No. 55,* 1966, pp. 503–536.
14. Taylor, E. and Davis, J. M., "Tidal Propagation and Dispersion in Estuaries," *Finite Elements in Fluids—Vol. 1,* R. H. Gallagher, J. T. Oden, C. Taylor, and O. C. Zienkiewicz, eds., John Wiley and Sons, New York, N.Y., 1975, pp. 95–118.
15. Wylie, C. R., Jr., *Advanced Engineering Mathematics,* 3rd ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1966.